REPORT DOCUMENTATION PAGE	Form Approved OMB NO. 0704-0188

1. REPORT DATE (DD-MM-YYYY)	2. REPORT TYPE	3. DATES COVERED (From - To)
08-08-2011	Conference Proceeding	-

4. TITLE AND SUBTITLE	
Sampling-Based Stochastic Sensitivity Analysis Using Score Functions for RBDO Problems with Correlated Random Variables	5a. CONTRACT NUMBER W911NF-09-1-0250
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER 611102
6. AUTHORS Kyung K. Choi, Yoojeong Noh, Liang Zhao, Ikjin Lee, David Gorsich	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAMES AND ADDRESSES	8. PERFORMING ORGANIZATION REPORT NUMBER
University of Iowa @ Iowa City Office of Sponsored Programs The University of Iowa Iowa City, IA 52242 -	

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
U.S. Army Research Office P.O. Box 12211 Research Triangle Park, NC 27709-2211	ARO
	11. SPONSOR/MONITOR'S REPORT NUMBER(S) 56025-NS.23

12. DISTRIBUTION AVAILIBILITY STATEMENT

Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

The views, opinions and/or findings contained in this report are those of the author(s) and should not contrued as an official Department of the Army position, policy or decision, unless so designated by other documentation.

14. ABSTRACT

This study presents a methodology for computing stochastic sensitivities with respect to the design variables, which are the mean values of the input correlated random variables. Assuming that an accurate surrogate model is available, the proposed method calculates the component reliability, system reliability, or statistical moments and their sensitivities by applying Monte Carlo simulation (MCS) to the accurate surrogate model. Since the surrogate model is used, the computational cost for the stochastic sensitivity analysis is negligible. The copula is used to

15. SUBJECT TERMS

Stochastic Sensitivity Analyses, Score Functions, Monte CarloSimulation, Copula, Surrogate Model, RBDO

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	15. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			Kyung Choi
UU	UU	UU	UU		19b. TELEPHONE NUMBER 319-335-5684

Report Title

Sampling-Based Stochastic Sensitivity Analysis Using Score Functions for RBDO Problems with Correlated Random Variables

ABSTRACT

This study presents a methodology for computing stochastic sensitivities with respect to the design variables, which are the mean values of the input correlated random variables. Assuming that an accurate surrogate model is available, the proposed method calculates the component reliability, system reliability, or statistical moments and their sensitivities by applying Monte Carlo simulation (MCS) to the accurate surrogate model. Since the surrogate model is used, the computational cost for the stochastic sensitivity analysis is negligible. The copula is used to model the joint distribution of the correlated input random variables, and the score function is used to derive the stochastic sensitivities of reliability or statistical moments for the correlated random variables. An important merit of the proposed method is that it does not require the gradients of performance functions, which are known to be erroneous when obtained from the surrogate model, or the transformation from X-space to U-space for reliability analysis. Since no transformation is required and the reliability or statistical moment is calculated in X-space, there is no approximation or restriction in calculating the sensitivities of the reliability or statistical moment. Numerical results indicate that the proposed method can estimate the sensitivities of the reliability or statistical moments very accurately, even when the input random variables are correlated.

Conference Name: ASME 2010 International Design Engineering Technical Conferences & Computers and Information in En

Conference Date: August 15, 2010

Proceedings of the ASME 2010 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference
IDETC/CIE 2010
August 15 – 18, 2010, Montreal, Canada

DETC2010-28591

Sampling-Based Stochastic Sensitivity Analysis Using Score Functions for RBDO Problems with Correlated Random Variables

Ikjin Lee, Kyung K. Choi, Yoojeong Noh, Liang Zhao
Department of Mechanical & Industrial Engineering, College of Engineering
The University of Iowa, Iowa City, IA 52242, USA
ilee@engineering.uiowa.edu, kkchoi@engineering.uiowa.edu, noh@engineering.uiowa.edu,
liazhao@engineering.uiowa.edu

David Gorsich
US Army RDECOM/TARDEC
Warren, MI 48397-5000, USA
david.gorsich@us.army.mil

ABSTRACT

This study presents a methodology for computing stochastic sensitivities with respect to the design variables, which are the mean values of the input correlated random variables. Assuming that an accurate surrogate model is available, the proposed method calculates the component reliability, system reliability, or statistical moments and their sensitivities by applying Monte Carlo simulation (MCS) to the accurate surrogate model. Since the surrogate model is used, the computational cost for the stochastic sensitivity analysis is negligible. The copula is used to model the joint distribution of the correlated input random variables, and the score function is used to derive the stochastic sensitivities of reliability or statistical moments for the correlated random variables. An important merit of the proposed method is that it does not require the gradients of performance functions, which are known to be erroneous when obtained from the surrogate model, or the transformation from X-space to U-space for reliability analysis. Since no transformation is required and the reliability or statistical moment is calculated in X-space, there is no approximation or restriction in calculating the sensitivities of the reliability or statistical moment. Numerical results indicate that the proposed method can estimate the sensitivities of the reliability or statistical moments very accurately, even when the input random variables are correlated.

KEYWORDS

Stochastic Sensitivity Analyses, Score Functions, Monte Carlo Simulation, Copula, Surrogate Model, RBDO

1. INTRODUCTION

Reliability-based design optimization (RBDO) and reliability-based robust design optimization (RBRDO) have been widely applied to various engineering applications such as stamping [1,2], vehicle design with durability [3,4], and noise, vibration, and harshness (NVH) analysis [5,6], where accurate sensitivities of performance functions are available. If accurate sensitivities are available in a complex physical system, then the most probable point (MPP)–based reliability analysis, which includes the First Order Reliability Method (FORM) [7-10], the Second Order Reliability Method (SORM) [11,12], and the MPP-based Dimension Reduction Method (DRM) [13-15], can be used for approximately assessing the reliability of the system, which is used as a probabilistic constraint in both RBDO and RBRDO. Furthermore, the first and second statistical moments of the performance function, which are used in the objective function of RBRDO, are approximated using the first-order Taylor series expansion [16-17] or the mean-based DRM [18-20].

However, for engineering applications where accurate sensitivities of performance functions are not available, such as drivetrain, crashworthiness, micro or nano mechanics, or fluid-structure interaction, the MPP-based reliability analysis, which uses the sensitivities of performance functions to find the MPP, cannot be directly used. Instead, surrogate models

have been widely used to carry out design optimization for the engineering applications where sensitivities are unavailable [21-23]. Once an accurate surrogate model is available for the design optimization, the direct Monte Carlo simulation (MCS) [24] can be applied to the accurate surrogate model to estimate the reliability or statistical moments of the system with negligible computational burden. The reliability or statistical moment will be called the probabilistic response in this paper. To use the probabilistic response obtained using the MCS for the design optimization, sensitivities of the probabilistic response are still required, which can be obtained using the finite difference method (FDM) or using the sensitivities of the surrogate model. However, since the probabilistic response is obtained from the MCS, the FDM may require a significant number of samples to obtain accurate sensitivities due to the statistical noise of the MCS. On the other hand, even if the surrogate model is very accurate for the response value, the sensitivities obtained from the surrogate model are known to be inaccurate, and accordingly it is not a good idea to use them for the design optimization.

The main objective of this paper is to propose a new sampling-based RBDO for correlated random inputs [25-27], which does not require obtaining the sensitivities of the performance functions and their sensitivities from the surrogate model. Instead, stochastic sensitivity analysis using the score function, which was proposed for the independent random variables [28] or correlated Gaussian random variables [29], is used to derive the sensitivities of the probabilistic response with correlated random variables. The proposed sensitivity analysis does not require the transformation from the original design space to the standard normal space, which means that there is no approximation or restriction in calculating the sensitivities of the probabilistic response. Since the generation of the accurate surrogate model is beyond the scope of this paper, it will not be explained here. For this, the dynamic Kriging (D-Kriging) method developed in Ref. 31 is used in this paper. Numerical examples demonstrate that the sensitivities developed in the paper agree very well with the sensitivities obtained from the FDM with 250 million samples for the MCS.

The paper is organized into four main parts. The first part, Section 2, explains how to mathematically express the component, the system probability of failure, and the statistical moments and their sensitivities. The second part, Section 3, shows how to derive the score function for statistically independent or correlated random variables. The third part, Section 4, shows the formulation of the sampling-based RBDO and RBRDO. The last part, Section 5, demonstrates with numerical examples the accuracy of the proposed sensitivities compared with the FDM results. Finally, the proposed method is integrated with the D-Kriging method to carry out RBDO and compared with four other RBDO methods to validate and point out the merit of the proposed method.

2. RELIABILITY, STATISTICAL MOMENTS AND SENSITIVITY

2.1 Reliability and Statistical Moments

A reliability analysis, for both the component and the system level, involves calculation of the probability of failure, denoted by P_F, which is defined using a multi-dimensional integral

$$P_F(\boldsymbol{\psi}) \equiv P[\mathbf{X} \in \Omega_F] = \int_{\mathbb{R}^N} I_{\Omega_F}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\psi}) d\mathbf{x} = E\left[I_{\Omega_F}(\mathbf{X})\right] \quad (1)$$

where $\boldsymbol{\psi}$ is a vector of distribution parameters, which usually includes the mean ($\boldsymbol{\mu}$) and/or standard deviation ($\boldsymbol{\sigma}$) of the random input $\mathbf{X} = \{X_1, \cdots, X_N\}^{\mathrm{T}}$, $P[\bullet]$ represents a probability measure, Ω_F is the failure set, $f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\psi})$ is a joint probability density function (PDF) of $\mathbf{X}$, and $E[\bullet]$ represents the expectation operator. The failure set is defined as $\Omega_F \equiv \{\mathbf{x} : G_i(\mathbf{x}) > 0\}$ for component reliability analysis of the i^{th} constraint function $G_i(\mathbf{x})$, and $\Omega_F \equiv \left\{\mathbf{x} : \bigcup_{i=1}^{NC} G_i(\mathbf{x}) > 0\right\}$ and $\Omega_F \equiv \left\{\mathbf{x} : \bigcap_{i=1}^{NC} G_i(\mathbf{x}) > 0\right\}$ for series system and parallel system reliability analysis of NC performance functions, respectively [29,30]. $I_{\Omega_F}(\mathbf{x})$ in Eq. (1) is called an indicator function and defined as

$$I_{\Omega_F}(\mathbf{x}) \equiv \begin{cases} 1, & \mathbf{x} \in \Omega_F \\ 0, & otherwise \end{cases} \quad (2)$$

In this paper, since the mean of $\mathbf{X}$, $\boldsymbol{\mu} = \{\mu_1, \cdots, \mu_N\}^{\mathrm{T}}$, is used as a design vector, the vector of distribution parameters $\boldsymbol{\psi}$ is simply replaced with $\boldsymbol{\mu}$ for the derivation of the sensitivity.

In a fashion similar to Eq. (1), the q^{th} statistical moment of a performance function $H(\mathbf{x})$ is defined as

$$m_q(\boldsymbol{\mu}) \equiv E[H^q(\mathbf{X})] = \int_{\mathbb{R}^N} H^q(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu}) d\mathbf{x}, \quad (3)$$

and thus, Eqs. (1) and (3) can be written in a generalized form as

$$h(\boldsymbol{\mu}) \equiv E[g(\mathbf{X})] = \int_{\mathbb{R}^N} g(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu}) d\mathbf{x} \quad (4)$$

which is called a probabilistic response [29]. In Eq. (4), $h(\boldsymbol{\mu})$ and $g(\mathbf{x})$ represent $P_F(\boldsymbol{\mu})$ and $I_{\Omega_F}(\mathbf{x})$, respectively, for reliability analysis, and $h(\boldsymbol{\mu})$ and $g(\mathbf{x})$ represent $m_q(\boldsymbol{\mu})$ and $H^q(\mathbf{x})$, respectively, for statistical moment analysis.

2.2 Stochastic Sensitivity Analysis

The sensitivity of the probabilistic response $h(\boldsymbol{\mu})$ with respect to μ_i is considered in this section. For the derivation of the sensitivity, the following four assumptions, which are known as the regularity conditions, are required [28,29].

1. The joint PDF $f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})$ is continuous.
2. The mean $\mu_i \in \mathrm{M}_i \subset \mathbb{R}$, $i = 1, \cdots, N$, where M_i is an open interval on $\mathbb{R}$.

3. The partial derivative $\frac{\partial f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})}{\partial \mu_i}$ exists and is finite for all $\mathbf{x}$ and μ_i. In addition, $h(\boldsymbol{\mu})$ is a differentiable function of $\boldsymbol{\mu}$.
4. There exists a Lebesgue integrable dominating function r(**x**) such that

$$\left| g(\mathbf{x}) \frac{\partial f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})}{\partial \mu_i} \right| \le r(\mathbf{x}) \tag{5}$$

for all **μ**.

With the four assumptions satisfied, taking the partial derivative of Eq. (4) with respect to μ_i yields

$$\frac{\partial h(\boldsymbol{\mu})}{\partial \mu_i} = \frac{\partial}{\partial \mu_i} \int_{\mathbb{R}^N} g(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu}) d\mathbf{x} \tag{6}$$

and the differential and integral operators can be interchanged due to Assumption 4 and the Lebesgue dominated convergence theorem [29,32], giving

$$\begin{aligned} \frac{\partial h(\boldsymbol{\mu})}{\partial \mu_i} &= \int_{\mathbb{R}^N} g(\mathbf{x}) \frac{\partial f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})}{\partial \mu_i} d\mathbf{x} \\ &= \int_{\mathbb{R}^N} g(\mathbf{x}) \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})}{\partial \mu_i} f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu}) d\mathbf{x} \\ &= E\left[g(\mathbf{x}) \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})}{\partial \mu_i} \right] \end{aligned} \tag{7}$$

since g(**x**) is not a function of μ_i. The partial derivative of the log function of the joint PDF in Eq. (7) with respect to μ_i is known as the first-order score function for μ_i and is denoted as

$$s_{\mu_i}^{(1)}(\mathbf{x};\boldsymbol{\mu}) \equiv \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})}{\partial \mu_i}. \tag{8}$$

Therefore, it is required to know the first-order score function to derive the sensitivity of the probabilistic response, which is either the reliability or the statistical moments. The derivation of the first-order score function for independent and correlated random variables will be shown in the subsequent section.

3. SENSITIVITY ANALYSIS BY SCORE FUNCTION

3.1 For Independent Random Variables

Consider a random input $\mathbf{X} = \{X_1, \cdots, X_N\}^{\mathrm{T}}$ whose components are statistically independent random variables. Then, the joint PDF of **X** is expressed as multiplication of its marginal PDFs as

$$f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu}) = \prod_{i=1}^{N} f_{X_i}(x_i;\mu_i) \tag{9}$$

where $f_{X_i}(x_i;\mu_i)$ is the marginal PDF corresponding to the i^{th} random variable X_i. Therefore, for statistically independent random variables, the first-order score function for μ_i is expressed as

$$s_{\mu_i}^{(1)}(\mathbf{x};\boldsymbol{\mu}) \equiv \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})}{\partial \mu_i} = \frac{\partial \ln f_{X_i}(x_i;\mu_i)}{\partial \mu_i}. \tag{10}$$

Since the marginal PDF and cumulative distribution function (CDF) are available analytically as listed in Table 1, where $\Phi(\bullet)$ and $\phi(\bullet)$ are the standard normal CDF and PDF, respectively, given by

$$\Phi(u) = \int_{-\infty}^{u} \phi(\xi) d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\left(-\frac{1}{2}\xi^2\right) d\xi, \tag{11}$$

the derivation of Eq. (10) is straightforward for normal, lognormal, and Gumbel distributions.

Table 1. Marginal PDF, CDF, and Its Parameters

	PDF, $f_X(x)$	CDF, $F_X(x)$	Parameters
Normal	$\frac{1}{\sqrt{2\pi}\sigma} e^{-0.5[\frac{x-\mu}{\sigma}]^2}$	$\Phi\left(\frac{x-\mu}{\sigma}\right)$	μ, σ
Log-normal	$\frac{1}{\sqrt{2\pi}x\bar{\sigma}} e^{-0.5[\frac{\ln x-\bar{\mu}}{\bar{\sigma}}]^2}$	$\Phi\left(\frac{\ln x-\bar{\mu}}{\bar{\sigma}}\right)$	$\bar{\sigma}^2 = \ln[1+(\frac{\sigma}{\mu})^2]$, $\bar{\mu} = \ln(\mu) - 0.5\bar{\sigma}^2$
Gumbel	$\alpha e^{-\alpha(x-\nu)-e^{-\alpha(x-\nu)}}$	$\alpha e^{-e^{-\alpha(x-\nu)}}$	$\mu = \nu + \frac{0.577}{\alpha}$, $\sigma = \frac{\pi}{\sqrt{6}\alpha}$
Weibull	$\frac{k}{\nu}(\frac{x}{\nu})^{k-1} e^{-(\frac{x}{\nu})^k}$	$1 - e^{-(\frac{x}{\nu})^k}$	$\mu = \nu\Gamma(1+\frac{1}{k})$, $\sigma^2 = \nu^2[\Gamma(1+\frac{2}{k}) - \Gamma^2(1+\frac{1}{k})]$
Uniform	$\frac{1}{b-a}$, $a \le x \le b$	$\frac{x-a}{b-a}$	$\mu = \frac{a+b}{2}$, $\sigma = \frac{b-a}{\sqrt{12}}$

However, for the case of Weibull distribution, the derivation is not straightforward since two distribution parameters, k and ν, are coupled as shown in Table 1. The score function for Weibull distribution is written as

$$\begin{aligned} s_{\mu_i}^{(1)}(\mathbf{x};\boldsymbol{\mu}) &= \frac{\partial}{\partial \mu_i}\left(\ln \frac{k_i}{\nu_i} + (k_i - 1)\ln \frac{x_i}{\nu_i} - (\frac{x_i}{\nu_i})^{k_i} \right) \\ &= \frac{1}{k_i}\frac{\partial k_i}{\partial \mu_i} - \frac{1}{\nu_i}\frac{\partial \nu_i}{\partial \mu_i} + \frac{\partial k_i}{\partial \mu_i}\ln\frac{x_i}{\nu_i} \\ &\quad - \frac{(k_i-1)}{\nu_i}\frac{\partial \nu_i}{\partial \mu_i} - (\frac{x_i}{\nu_i})^{k_i}\left(\frac{\partial k_i}{\partial \mu_i}\ln\frac{x_i}{\nu_i} - \frac{k_i}{\nu_i}\frac{\partial \nu_i}{\partial \mu_i} \right) \end{aligned} \tag{12}$$

which requires the calculation of $\frac{\partial k_i}{\partial \mu_i}$ and $\frac{\partial \nu_i}{\partial \mu_i}$ since k_i and ν_i are functions of μ_i. As shown in Table 1, both $\frac{\partial k_i}{\partial \mu_i}$ and $\frac{\partial \nu_i}{\partial \mu_i}$ require the evaluation of the gamma function Γ defined as

$\Gamma(z)=\int_0^{\infty} t^{z-1}e^{-t}dt$ and its derivative. By taking the partial derivative on two parameters of Weibull distribution and using the derivative of the gamma function, we can obtain $\frac{\partial k_i}{\partial \mu_i}$ and $\frac{\partial \nu_i}{\partial \mu_i}$ as

$$\begin{bmatrix} \frac{\partial \nu_i}{\partial \mu_i} \\ \frac{\partial k_i}{\partial \mu_i} \end{bmatrix} = \begin{bmatrix} \Gamma\left(1+\frac{1}{k_i}\right) & -\frac{\nu_i}{k_i^2}\int_0^{\infty} t^{\frac{1}{k_i}} e^{-t}\ln t dt \\ 2\nu_i\Gamma\left(1+\frac{2}{k_i}\right) & -\frac{2\nu_i^2}{k_i^2}\int_0^{\infty} t^{\frac{2}{k_i}} e^{-t}\ln t dt \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2\mu_i \end{bmatrix} \tag{13}$$

and inserting Eq. (13) into Eq. (12) yields the first-order score function for Weibull distribution. Table 2 summarizes the first-order score functions for four marginal PDFs. $\frac{\partial \bar{\mu}_i}{\partial \mu_i}$ and $\frac{\partial \bar{\sigma}_i}{\partial \mu_i}$ for lognormal distribution in Table 2 can be easily derived using the definition of $\bar{\mu}_i$ and $\bar{\sigma}_i$ shown in Table 1, and $\frac{\partial k_i}{\partial \mu_i}$ and $\frac{\partial \nu_i}{\partial \mu_i}$ for Weibull distribution in Table 2 are shown in Eq. (13).

Table 2. First-Order Score Function for μ_i for Independent Random Variables

Marginal Distribution	First-Order Score Function, $s_{\mu_i}^{(1)}(\mathbf{x};\boldsymbol{\mu})$
Normal	$\frac{x_i-\mu_i}{\sigma_i^2}$
Log-normal	$-\frac{1}{\bar{\sigma}_i}\frac{\partial \bar{\sigma}_i}{\partial \mu_i}+\frac{1}{\bar{\sigma}_i^2}\left(\frac{\ln x_i-\bar{\mu}_i}{\bar{\sigma}_i}\right)\times\left[\bar{\sigma}_i\frac{\partial \bar{\mu}_i}{\partial \mu_i}+(\ln x_i-\bar{\mu}_i)\frac{\partial \bar{\sigma}_i}{\partial \mu_i}\right]$
Gumbel	$\alpha_i-\alpha_i e^{-\alpha_i(x_i-\nu_i)}$
Weibull	$\frac{1}{k_i}\frac{\partial k_i}{\partial \mu_i}-\frac{1}{\nu_i}\frac{\partial \nu_i}{\partial \mu_i}+\frac{\partial k_i}{\partial \mu_i}\ln\frac{x_i}{\nu_i}-\frac{(k_i-1)}{\nu_i}\frac{\partial \nu_i}{\partial \mu_i} -(\frac{x_i}{\nu_i})^{k_i}\left(\frac{\partial k_i}{\partial \mu_i}\ln\frac{x_i}{\nu_i}-\frac{k_i}{\nu_i}\frac{\partial \nu_i}{\partial \mu_i}\right)$

Finally, the uniform distribution, as shown in Table 1, is not continuous on $\mathbb{R}$, which makes the joint PDF in Eq. (9) discontinuous. This discontinuity of the uniform distribution violates the first assumption explained in Section 2.2. Thus, the score function cannot be used if a random variable follows uniform distribution.

Assuming X_i follows the uniform distribution and all the components of $\mathbf{X}$ are statistically independent random variables, then the sensitivity of the probabilistic response with respect to μ_i shown in Eq. (6) can be written as

$$\frac{\partial h(\boldsymbol{\mu})}{\partial \mu_i}=\frac{\partial}{\partial \mu_i}\int_{\mathbb{R}^{N-1}}\int_{a(\mu_i)}^{b(\mu_i)} g(\mathbf{x}) f_{X_i}(x_i;\mu_i) f_{\tilde{\mathbf{X}}}(\tilde{\mathbf{x}};\tilde{\boldsymbol{\mu}})dx_i d\tilde{\mathbf{x}} \tag{14}$$

where $\tilde{\mathbf{X}}$ and $\tilde{\boldsymbol{\mu}}$ are vectors of the random variables and its means, respectively, without the i^{th} component. Since two integral limits, a and b, are functions of the differential variable μ_i, the differential and integral operators cannot be interchanged directly. Instead, the Leibniz integral rule [33] gives a formula for differentiation of a definite integral whose limits are functions of the differential variable, which is given by

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x,z)dx=\int_{a(z)}^{b(z)}\frac{\partial f}{\partial z}dx+f(b(z),z)\frac{\partial b}{\partial z} - f(a(z),z)\frac{\partial a}{\partial z} \tag{15}$$

Applying the Leibniz integral rule to Eq. (14) yields

$$\begin{aligned}\frac{\partial h(\boldsymbol{\mu})}{\partial \mu_i} &= \frac{\partial}{\partial \mu_i}\int_{\mathbb{R}^{N-1}}\frac{g(\tilde{\mathbf{x}},b)-g(\tilde{\mathbf{x}},a)}{b-a} f_{\tilde{\mathbf{X}}}(\tilde{\mathbf{x}};\tilde{\boldsymbol{\mu}})d\tilde{\mathbf{x}} \\ &= E\left[\frac{g(\tilde{\mathbf{X}},b)-g(\tilde{\mathbf{X}},a)}{b-a}\right]\end{aligned} \tag{16}$$

which is the sensitivity of the probabilistic response with respect to μ_i when X_i follows uniform distribution.

3.2 For Correlated Input Random Variables

Consider a bivariate correlated random input $\mathbf{X}=\left\{X_i, X_j\right\}^{\mathrm{T}}$. Then, the joint PDF of $\mathbf{X}$ is expressed as [26,27]

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu}) &= \frac{\partial^2 C(u,v;\theta)}{\partial u \partial v} f_{X_i}(x_i;\mu_i) f_{X_j}(x_j;\mu_j) \\ &= C_{,uv}(u,v;\theta) f_{X_i}(x_i;\mu_i) f_{X_j}(x_j;\mu_j)\end{aligned} \tag{17}$$

where C is a copula function, $u=F_{X_i}(x_i;\mu_i)$ and $v=F_{X_j}(x_j;\mu_j)$ are CDFs for X_i and X_j, respectively, and θ is the correlation coefficient. Table 3 lists commonly used copula functions [26,27].

Table 3. Commonly Used Copula Functions

Copula Type	Copula Function, $C(u,v\|\theta)$
Clayton	$\left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta}$
AMH	$\frac{uv}{1-\theta(1-u)(1-v)}$
Frank	$-\frac{1}{\theta}\ln\left[1+\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)/\left(e^{-\theta}-1\right)\right]$
FGM	$uv+\theta uv(1-u)(1-v)$
Gaussian	$\int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)}\frac{1}{2\pi\sqrt{1-\theta^2}}\exp\left(\frac{2\theta sw-s^2-w^2}{2(1-\theta^2)}\right)dsdw$
Independent	uv

The partial derivative of the copula function with respect to the marginal CDFs u and v is called the copula density function and denoted as

$$c(u,v\mid\theta)\equiv\frac{\partial^2 C(u,v;\theta)}{\partial u\partial v}=C_{,uv}(u,v;\theta) \tag{18}$$

and displayed in Table 4. As shown in Table 4, the joint PDF for independent random variables explained in Eq. (9) is a special case of Eq. (17) where the independent copula function is used.

Table 4. Copula Density Functions

Copula Type	Copula Density Functions, $c(u,v;\theta)$
Clayton	$(1+\theta)(uv)^{-(1+\theta)}\left(-1+u^{-\theta}+v^{-\theta}\right)^{-\left(2+\frac{1}{\theta}\right)}$
AMH	$\dfrac{1+\theta^2(1-u)(1-v)-\theta(2-u-v-uv)}{\left[1-\theta(1-u)(1-v)\right]^3}$
Frank	$\dfrac{\theta e^{\theta(1+u+v)}\left(e^{\theta}-1\right)}{\left\{e^{\theta}-e^{\theta(1+u)}-e^{\theta(1+v)}+e^{\theta(u+v)}\right\}^2}$
FGM	$1+\theta(1-2u)(1-2v)$
Gaussian	$\dfrac{1}{\sqrt{1-\theta^2}}\exp\left(\dfrac{\Phi^{-1}(u)^2+\Phi^{-1}(v)^2}{2}\right)\times \exp\left(\dfrac{2\theta\Phi^{-1}(u)\Phi^{-1}(v)-\Phi^{-1}(u)^2-\Phi^{-1}(v)^2}{2(1-\theta^2)}\right)$
Independent	1

Accordingly, using Eq. (17), the first-order score functions in Eq. (8) for a correlated bivariate input are expressed as

$$s_{\mu_i}^{(1)}(\mathbf{x};\boldsymbol{\mu})\equiv\frac{\partial\ln f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})}{\partial\mu_i}=\frac{\partial\ln c(u,v;\theta)}{\partial\mu_i}+\frac{\partial\ln f_{X_i}(x_i;\mu_i)}{\partial\mu_i}\,. \tag{19}$$

The derivation of the first term of the right-hand side of Eq. (19) is straightforward using Table 4 and listed in Table 5, and the second term of the right-hand side of Eq. (19) is identical to Eq. (10), so it can be obtained from Table 2. One can see from Table 5 that Eq. (19) is identical to Eq. (10) if the independent copula is used. In Table 5, the partial derivative of the marginal CDF with respect to μ_i, $\dfrac{\partial u}{\partial\mu_i}$ is easily obtained from Table 1 and is shown in Table 6.

Table 5. Log-derivative of Copula Density Function

Copula Type	$\dfrac{\partial\ln c(u,v;\theta)}{\partial\mu_i}$
Clayton	$\left(-\dfrac{1+\theta}{u}+\dfrac{(2\theta+1)u^{-(1+\theta)}}{u^{-\theta}+v^{-\theta}-1}\right)\dfrac{\partial u}{\partial\mu_i}$
AMH	$\left[\dfrac{\theta^2(1-v)+\theta(v+1)}{1-\theta^2(1-u)(1-v)-\theta(2-u-v-uv)}-\dfrac{3\theta(1-v)}{1-\theta(1-u)(1-v)}\right]\dfrac{\partial u}{\partial\mu_i}$
Frank	$\theta\left[\dfrac{2\left(e^{\theta(1+u)}-e^{\theta(u+v)}\right)}{e^{\theta}-e^{\theta(1+u)}-e^{\theta(1+v)}+e^{\theta(u+v)}}+1\right]\dfrac{\partial u}{\partial\mu_i}$
FGM	$\left[\dfrac{2\theta(2v-1)}{1+\theta(1-2u)(1-2v)}\right]\dfrac{\partial u}{\partial\mu_i}$
Gaussian	$\left[\dfrac{\Phi^{-1}(u)}{\phi(\Phi^{-1}(u))}+\dfrac{\theta\Phi^{-1}(v)-\Phi^{-1}(u)}{\phi(\Phi^{-1}(u))(1-\theta^2)}\right]\dfrac{\partial u}{\partial\mu_i}$
Independent	0

Table 6. Partial Derivatives of Marginal Distribution with respect to μ_i

Marginal Distribution	Partial Derivatives of Marginal Distribution, $\dfrac{\partial u}{\partial\mu_i}$
Normal	$\displaystyle\int_{-\infty}^{x_i}\frac{1}{\sqrt{2\pi}\sigma_i}\exp\left[-\frac{1}{2}\left(\frac{\xi-\mu_i}{\sigma_i}\right)^2\right]\times\frac{\xi-\mu_i}{\sigma_i^2}d\xi$
Log-normal	$\displaystyle\int_{-\infty}^{x_i}\frac{e^{-0.5[\frac{\ln\xi-\bar{\mu}_i}{\bar{\sigma}_i}]^2}}{\sqrt{2\pi}\bar{\sigma}_i^2\xi}\left[-\frac{\partial\bar{\sigma}_i}{\partial\mu_i}+\frac{\ln\xi-\bar{\mu}_i}{\bar{\sigma}_i}\frac{\partial\bar{\mu}_i}{\partial\mu_i}+\frac{(\ln\xi-\bar{\mu}_i)^2}{\bar{\sigma}_i^2}\frac{\partial\bar{\sigma}_i}{\partial\mu_i}\right]d\xi$
Gumbel	$-\alpha_i e^{-\alpha_i(x_i-\nu_i)-e^{-\alpha_i(x_i-\nu_i)}}$
Weibull	$e^{-(\frac{x_i}{\nu_i})^{k_i}}\left(\dfrac{x_i}{\nu_i}\right)^{k_i}\left(\dfrac{\partial k_i}{\partial\mu_i}\ln\dfrac{x_i}{\nu_i}-\dfrac{k_i}{\nu_i}\dfrac{\partial\nu_i}{\partial\mu_i}\right)$
Uniform	$-\dfrac{1}{b-a}$

In a similar way to that explained in Section 3.1, consider a bivariate correlated random input $\mathbf{X}=\left\{X_i,X_j\right\}^{\mathrm{T}}$ where X_i is assumed to follow the uniform distribution. Then, the sensitivity of the probabilistic response with respect to μ_i for the bivariate correlated random input can be written as

$$\frac{\partial h(\boldsymbol{\mu})}{\partial\mu_i}=\frac{\partial}{\partial\mu_i}\int_{\mathbb{R}}\int_{a(\mu_i)}^{b(\mu_i)}g(\mathbf{x})c(u,v;\theta)f_{X_i}(x_i;\mu_i)f_{X_j}(x_j;\mu_j)dx_i dx_j \tag{20}$$

Applying the Leibniz integral rule to Eq. (20) yields

$$\begin{aligned}\frac{\partial h(\boldsymbol{\mu})}{\partial\mu_i}=E[&\int_{a(\mu_i)}^{b(\mu_i)}\frac{g(X_j,x_i)}{b-a}\frac{\partial c(u,v;\theta)}{\partial\mu_i}dx_i\\&+\frac{g(X_j,b)c(X_j,b)-g(X_j,a)c(X_j,a)}{b-a}]\end{aligned} \tag{21}$$

and Eq. (21) is identical with Eq. (16) if the independent copula is used since $c(u,v;\theta)=1$.

4. For Both Independent and Correlated Input Random Variables

Consider a random input $\mathbf{X}=\left\{X_1,\cdots,X_N\right\}^{\mathrm{T}}$ where M pairs of bivariate correlated random variables exist. Then, the joint PDF of $\mathbf{X}$ is expressed as

$$f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})=\prod_{j=1}^{M}c_j(u,v;\theta_j)\prod_{i=1}^{N}f_{X_i}(x_i;\mu_i) \tag{22}$$

from Eqs. (9) and (17). Taking the partial derivative on both sides of Eq. (22) yields the first-order score functions for a general random input **X** as

$$\begin{aligned} s_{\mu_i}^{(1)}(\mathbf{x};\boldsymbol{\mu}) &\equiv \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})}{\partial \mu_i} \\ &= \frac{\partial \ln c_j(u,v;\theta_j)}{\partial \mu_i} + \frac{\partial \ln f_{X_i}(x_i;\mu_i)}{\partial \mu_i}. \end{aligned} \tag{23}$$

Thus, if X_i is a correlated random variable, Eq. (23) is identical to Eq. (19), and if X_i is statistically independent, Eq. (23) is identical to Eq. (10). Similarly, if X_i follows the uniform distribution, Eq. (21) can be used to calculate the sensitivity of the probabilistic response with a general random input **X**.

5. FORMULATION OF SAMPLING-BASED RBDO

The mathematical formulation of a general RBDO problem is expressed as

$$\begin{aligned} &\text{minimize} && \text{Cost}(\mathbf{d}) \\ &\text{subject to} && P[G_i(\mathbf{X})>0] \le P_{F_i}^{\text{Tar}},\ i=1,\cdots,NC \\ & && \mathbf{d}^{\text{L}} \le \mathbf{d} \le \mathbf{d}^{\text{U}},\quad \mathbf{d}\in \text{R}^{ndv} \text{ and } \mathbf{X}\in \text{R}^{nrv} \end{aligned} \tag{24}$$

where $\mathbf{d}=\{d_i\}^{\text{T}}=\boldsymbol{\mu}(\mathbf{X})$ is the design vector, which is the mean value of the *N*-dimensional random vector $\mathbf{X}=\{X_1, X_2,\cdots, X_N\}^{\text{T}}$; $P_{F_i}^{\text{Tar}}$ is the target probability of failure for the i^{th} constraint; and *NC*, *ndv*, and *nrv* are the number of probabilistic constraints, design variables, and random variables, respectively. The mathematical formulation of a general RBRDO is given by

$$\begin{aligned} &\text{minimize} && f(\mu_H,\sigma_H^2) \\ &\text{subject to} && P[G_i(\mathbf{X})>0] \le P_{F_i}^{\text{Tar}},\ i=1,\cdots,NC \\ & && \mathbf{d}^{\text{L}} \le \mathbf{d} \le \mathbf{d}^{\text{U}},\quad \mathbf{d}\in \text{R}^{ndv} \text{ and } \mathbf{X}\in \text{R}^{nrv} \end{aligned} \tag{25}$$

where μ_H and σ_H^2 are the mean value and variance of the performance function H(**X**), respectively.

To carry out RBDO and RBRDO using Eqs. (24) and (25), respectively, it is required to know the function value and its sensitivities at a given design. However, in most engineering applications, it is very difficult, if not impossible, to obtain accurate sensitivities. For engineering applications where accurate sensitivities are not available, surrogate models have been widely used to carry out design optimization. Once an accurate surrogate model is available for the design optimizations, the MCS can be applied to estimate the reliability or statistical moments of the system with negligible computational burden.

Denote the surrogate models for constraint function $G_i(\mathbf{X})$ and performance function H(**X**) as $\hat{G}_i(\mathbf{X})$ and $\hat{H}(\mathbf{X})$, respectively. Then, by applying the MCS to the surrogate model $\hat{G}_i(\mathbf{X})$, the probabilistic constraints in Eqs. (24) and (25) can be approximated as

$$P_{F_i} \equiv P[G_i(\mathbf{X})>0] \cong \frac{1}{K}\sum_{k=1}^{K} I_{\hat{\Omega}_F}(\mathbf{x}^{(k)}) \le P_{F_i}^{\text{Tar}} \tag{26}$$

where K is the MCS sample size, $\mathbf{x}^{(k)}$ is the k^{th} realization of **X**, and the failure set $\hat{\Omega}_F$ for the surrogate model is defined as $\hat{\Omega}_F \equiv \left\{\mathbf{x}: \hat{G}_i(\mathbf{x})>0\right\}$. Sensitivity of the probabilistic constraint in Eqs. (24) and (25) is obtained using the score function explained in Section 3 as

$$\frac{\partial P_{F_i}}{\partial \mu_i} \cong \frac{1}{K}\sum_{k=1}^{K} I_{\hat{\Omega}_F}(\mathbf{x}^{(k)})\, s_{\mu_i}^{(1)}(\mathbf{x}^{(k)};\boldsymbol{\mu}) \tag{27}$$

where $s_{\mu_i}^{(1)}(\mathbf{x}^{(k)};\boldsymbol{\mu})$ is obtained using Eq. (23).

In a similar manner, the statistical moments in Eq. (25) can be approximated as

$$\mu_{\hat{H}} \cong \frac{1}{K}\sum_{k=1}^{K}\hat{H}(\mathbf{x}^{(k)}) \text{ and } \sigma_{\hat{H}}^2 \cong \frac{1}{K}\sum_{k=1}^{K}\hat{H}^2(\mathbf{x}^{(k)}) - \mu_{\hat{H}}^2, \tag{28}$$

respectively, and their sensitivities are

$$\frac{\partial \mu_{\hat{H}}}{\partial \mu_i} \cong \frac{1}{K}\sum_{k=1}^{K}\hat{H}(\mathbf{x}^{(k)})\, s_{\mu_i}^{(1)}(\mathbf{x}^{(k)};\boldsymbol{\mu}) \tag{29}$$

and

$$\frac{\partial \sigma_{\hat{H}}^2}{\partial \mu_i} \cong \frac{1}{K}\sum_{k=1}^{K}\hat{H}^2(\mathbf{x}^{(k)})\, s_{\mu_i}^{(1)}(\mathbf{x}^{(k)};\boldsymbol{\mu}) - 2\mu_{\hat{H}}\frac{\partial \mu_{\hat{H}}}{\partial \mu_i}, \tag{30}$$

respectively.

As shown in Eqs. (27), (29), and (30), the sensitivity calculation using the score function and MCS does not require the sensitivity of the surrogate models, which is known to be inaccurate even if the surrogate model accurately approximates the function values. Furthermore, the computation of the sensitivity using the score function does not include any approximation except the statistical noise due to the MCS, which can be avoided using a sufficiently large MCS sample size. In addition, this sensitivity analysis does not require the transformation from the original design space to the standard normal space, which usually makes the performance function become highly nonlinear, especially when the random input follows non-Gaussian marginal distribution and is correlated. Therefore, the sensitivity analysis using the score function and MCS will be very accurate and computationally efficient for engineering applications with correlated random input once accurate surrogate models are available; this will be verified through numerical examples in the subsequent section.

6. NUMERICAL EXAMPLES

Numerical studies are carried out in this section to verify the stochastic sensitivities derived in Section 3 using the score function. For the benchmark sensitivity to test the accuracy of the proposed method, the FDM using the MCS with 250

million samples is used. The stochastic sensitivities of the component probability of failure and statistical moments are compared with the FDM results in Section 5.1 and Section 5.2, respectively. Section 5.3 illustrates how the proposed sensitivity combined with an accurate surrogate model can be used to solve an RBDO problem.

6.1 Sensitivities of Component Probability of Failure

Consider a 2-D highly nonlinear polynomial function, which was studied in Ref. 15,

$$G_1(\mathbf{X}) = 0.7361 + (Y-6)^2 + (Y-6)^3 - 0.6\times(Y-6)^4 + Z \quad (31)$$

where $\begin{Bmatrix} Y \\ Z \end{Bmatrix} = \begin{bmatrix} 0.9063 & 0.4226 \\ 0.4226 & -0.9063 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$. As shown in Table 7, X_1 and X_2 follow Weibull and normal distributions, respectively. The Weibull distribution with the scale parameter of 4 and shape parameter of 15 has a mean of 3.8627 and a standard deviation of 0.3159. Two random variables X_1 and X_2 are assumed to be correlated with each copula shown in Table 3. Except for the independent copula case where the correlation coefficient is always zero, the correlation of coefficient of τ=0.3 is used. For the sensitivity calculation of the component probability of failure using the score function, the sample size for the MCS is 1 million, whereas the sensitivity calculation using the FDM employs 250 million samples for the MCS. This is because the FDM uses the difference of two probabilities of failure obtained from the MCS, which could include relatively large statistical noise if a small number of samples are used.

Table 7. Properties of Random Variables

Random Variables	Distribution	Distribution Parameters	
X_1	Weibull	v=4	k=15
X_2	Normal	μ=2.5	σ=0.3

Tables 8 and 9 show the comparison of sensitivities with respect to the mean of the non-normal random variable X_1 and the normal random variable X_2, respectively. Both tables illustrate that the sensitivities of the probability of failure obtained using the score function in Eq. (27) and MCS with 1 million samples agree very well with the sensitivities obtained using the FDM and MCS with 250 million samples. As mentioned previously, this good agreement is very obvious since the sensitivity derivation explained in Section 3 does not include any approximation except the statistical noise due to the MCS. Hence, regardless of the marginal distribution and copula types of random variables, the stochastic sensitivity using the score function is very accurate, and it is also computationally efficient since the MCS uses a relatively small sample size compared with the sensitivity analysis using the FDM, and no finite difference perturbation size is involved.

Table 8. Sensitivity of Probability of Failure w.r.t μ_1

Copula	Sensitivity	
	Score Function	FDM with perturbation size 0.1%
Clayton	9.7944	9.7446
AMH	9.7148	9.7354
Frank	8.9506	8.9208
FGM	9.7167	9.7421
Gaussian	7.4054	7.4158
Independent	9.4483	9.4296

Table 9. Sensitivity of Probability of Failure w.r.t μ_2

Copula	Sensitivity	
	Score Function	FDM with perturbation size 0.1%
Clayton	0.6485	0.6513
AMH	0.6701	0.6651
Frank	0.8149	0.8321
FGM	0.9318	0.9394
Gaussian	0.8347	0.8165
Independent	0.0897	0.0884

To verify whether the proposed stochastic sensitivity analysis works for high-dimensional problems, consider a 4-D polynomial function

$$\begin{aligned} G_2(\mathbf{X}) = 145 - 11X_1 + X_1^2 + 2X_2^2 - 20X_2 \\ -10X_3 + X_3^2 - 21X_4 + 2X_4^2 \end{aligned} \quad (32)$$

The properties of the random variables in Eq. (32) are shown in Table 10. For this problem, two random variables X_1 and X_2 are assumed to be correlated with each copula shown in Table 3, and two random variables X_3 and X_4 are assumed to be statistically independent. The relatively large correlation coefficient (τ=0.7) for X_1 and X_2 is used except for the independent copula. However, since the AMH copula cannot deal with a correlation coefficient larger than 1/3 [27], the AMH copula is excluded in this example.

Table 10. Properties of Random Variables

Random Variables	Distribution	Mean	Standard Deviation
X_1	Normal	5.0	0.3
X_2	Normal	6.0	0.3
X_3	Normal	7.0	0.3
X_4	Normal	4.0	0.3

Table 11 compares two sensitivities of the probability of failure with respect to μ_1. As in the previous example, two sensitivities agree very well with each other. Accordingly, it can be said that the proposed stochastic sensitivity analysis for the probability of failure is very accurate, regardless of the type of performance functions and copulas.

Table 11. Sensitivity of Probability of Failure w.r.t μ_1

Copula	Sensitivity	
	Score Function	FDM with perturbation size 0.1%
Clayton	-0.5719	-0.5643
Frank	-0.4925	-0.4862

FGM	-0.6605	-0.6712
Gaussian	-0.6005	-0.5964
Independent	-1.5890	-1.5824

6.2 Sensitivities of Statistical Moments

Consider the same 2-D highly nonlinear polynomial function shown in Eq. (31) for the sensitivity comparison of statistical moments. X_1 and X_2 follow Weibull and normal distributions, respectively, with the same distribution parameters as shown in Table 7. The sensitivities of the first two statistical moments, mean and variance, of the performance function in Eq. (31) are obtained using Eqs. (29) and (30), respectively. Again, 1 million MCS samples are used for the sensitivity calculation using Eqs. (29) and (30), that is, K=1,000,000. For the sensitivity of the statistical moment using the FDM, the statistical moment is obtained first using the MCS with 250 million samples. Then, by perturbing the mean of X_1 by 0.1%, the perturbed statistical moment is obtained, and the sensitivity is obtained using the difference between two statistical moments. So, for a problem with two random variables as in Eq. (31), the sensitivity analysis using the FDM requires three MCSs with 250 million samples. The computational time for the sensitivity analysis using the FDM is 750 times that of the proposed sensitivity analysis, and this difference will increase as the number of random variables increases.

Table 12 compares two sensitivities of statistical moments with respect to μ_1. Regardless of the copula type used, sensitivities of two statistical moments obtained using the score function and FDM agree very well with each other.

Table 12. Sensitivity of Statistical Moments with respect to μ_1

Copula	Mean		Variance	
	Score Function	FDM with perturbation size 0.1%	Score Function	FDM with perturbation size 0.1%
Clayton	11.8981	11.6201	-140.2574	-136.1142
Frank	11.8093	11.8681	-109.1958	-107.4361
FGM	11.2145	11.1848	-105.3351	-104.3938
Gaussian	11.8488	11.9547	-122.4572	-124.6348
Independent	11.2221	11.2120	-71.5755	-71.1807

6.3 Sampling-Based RBDO using Proposed Stochastic Sensitivity

To see how the proposed sensitivity analysis works for an RBDO problem, consider a 2-D mathematical RBDO problem, which is formulated to

$$\begin{aligned}
&\text{minimize} && C(\mathbf{d}) = -\frac{(d_1+d_2-10)^2}{30} - \frac{(d_1-d_2+10)^2}{120} \\
&\text{subject to} && P(G_i(\mathbf{X}(\mathbf{d}))>0) \le P_{F_i}^{\text{Tar}} = 2.275\%, \quad i=1\sim 3 \\
& && \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U, \quad \mathbf{d}\in \mathrm{R}^2 \text{ and } \mathbf{X}\in \mathrm{R}^2
\end{aligned} \tag{33}$$

where three constraints are given by

$$\begin{aligned}
G_1(\mathbf{X}) &= 1 - \frac{X_1^2 X_2}{20} \\
G_2(\mathbf{X}) &= -1 + (Y-6)^2 + (Y-6)^3 - 0.6\times(Y-6)^4 + Z \\
G_3(\mathbf{X}) &= 1 - \frac{80}{X_1^2 + 8X_2 + 5}
\end{aligned} \tag{34}$$

where $\begin{Bmatrix} Y \\ Z \end{Bmatrix} = \begin{bmatrix} 0.9063 & 0.4226 \\ 0.4226 & -0.9063 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, which are drawn in Figure 1. The properties of two random variables are shown in Table 13, and they are assumed to be correlated with the Clayton copula (τ=0.5). As shown in Eq. (33), the target probability of failure (P_F^{Tar}) is 2.275% for all constraints.

Table 13. Properties of Random Variables

Random Variables	Distribution	d^L	d^O	d^U	Standard Deviation
X_1	Normal	0.0	5.0	10.0	0.3
X_2	Normal	0.0	5.0	10.0	0.3

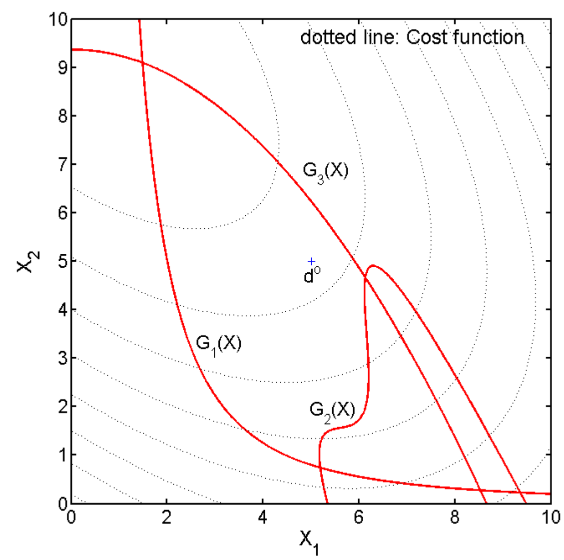

Figure 1. Shape of Constraint and Cost Functions

To apply the proposed sensitivity analysis to an RBDO problem, accurate surrogate models need to be utilized. For that purpose, the D-Kriging method with sequential sampling proposed by Zhao et al. [31] is used because, in terms of accuracy, the D-Kriging method outperforms existing methods such as the polynomial response surface method, the radial basis function (RBF) method, the support vector regression (SVR) method, and the universal Kriging method. Assuming the constraint functions in Eq. (34) are not given analytically, which is usually described as the black-box constraint, the surrogate models for the constraint functions are first generated using the D-Kriging method. Then, using the generated surrogate models, the probabilities of failure for three constraints and their sensitivities are obtained using the MCS with 1 million samples.

Table 14 compares the numerical results of five different RBDO methods. The first three results are obtained from the so-called MPP-based RBDO, which requires the sensitivity of the constraint functions for the MPP search and design optimization. This MPP-based RBDO includes the FORM, the DRM with three quadrature points, which is denoted as DRM3 in Table 14, and the DRM with five quadrature points, which is denoted as DRM5. The last two results are obtained from the sampling-based RBDO, which uses the MCS for the estimation of the probability of failure and its sensitivity. The sampling-based RBDO using the D-Kriging method is the proposed method, and for the comparison of the accuracy of the proposed method, the result of the sampling-based RBDO using the true functions given in Eq. (34) is also shown in the table.

Table 14. Comparison of Various RBDOs ($P_F^{Tar} = 2.275\%$)

Methods		Cost	Optimum Design	MCS		Function Call
				P_{F_1} ,%	P_{F_2} ,%	
MPP-Based RBDO	FORM	-1.8742	5.0026, 1.6165	2.3022	1.2835	52+52
	DRM3	-1.8794	5.0315, 1.6050	2.2912	1.7496	128+106
	DRM5	-1.8821	5.0454, 1.5988	2.2621	2.0183	146+102
Sampling-Based RBDO	D-Kriging	-1.8864	5.0595, 1.5893	2.2841	2.3056	57
	True Function	-1.8853	5.0541, 1.5918	2.2912	2.2791	N.A.

From the table, it can be seen that the probability of failure of the second constraint (1.2835%) at the optimum design obtained using the FORM is not close to the target probability of failure (2.275%). This is because the second constraint is highly nonlinear as shown in Figure 1 and the FORM cannot accurately estimate the probability of failure of highly nonlinear functions. To improve the accuracy of the probability of failure at the optimum design, the MPP-based DRM with three or five quadrature points can be used; Table 14 shows that the MPP-based DRM indeed improves the accuracy of the probability of failure at the optimum design. However, to obtain a more accurate optimum design, more quadrature points are required, such as the DRM7, etc. The FORM uses 52 function evaluations and 52 sensitivity calculations, whereas the MPP-based DRM with five quadrature points uses 146 function evaluations and 102 sensitivity calculations to obtain the optimum design, and the number of function evaluations for the MPP-based DRM will be increased as the number of quadrature points increases.

On the other hand, the sampling-based RBDO, which uses the D-Kriging method and the proposed stochastic sensitivity analysis, shows very accurate optimum design; yet it requires only 57 samples for the accurate optimum design. Without the sensitivity of the performance functions, the sampling-based RBDO can obtain a very accurate optimum design and the optimum design is very close to the optimum design obtained using the true functions. This means that the D-Kriging method generates very accurate surrogate models for the true functions. From this example, it can be said that once accurate surrogate models are available, the sampling-based RBDO using the proposed sensitivity analysis yields very accurate optimum designs with good efficiency.

More detailed discussion on the sampling-based RBDO, which includes higher dimensional engineering applications such as M1A1 Abrams tank roadarm [15], needs to be carried out and is ongoing.

7. CONCLUSIONS

The stochastic sensitivity analysis of the probabilistic constraints and statistical moments with respect to mean values of correlated random variables using the score function and MCS is carried out in this study. Since it does not require the sensitivity of performance functions or even the sensitivity of surrogate models, the proposed sensitivity analysis yields very accurate sensitivity estimation regardless of the marginal and copula types of the random input once accurate surrogate models are available. Furthermore, the proposed method uses only one MCS at a given design to obtain the probability of failure and its sensitivity or statistical moments and their sensitivities simultaneously, whereas the FDM uses N+1 MCS to obtain the sensitivities of the probabilistic response, where N is the number of random variables. Thus, the proposed sensitivity analysis using the score function is far more efficient than the FDM. In addition, the proposed sensitivity analysis is more accurate than the FDM when the same MCS sample size is used. Hence, the proposed stochastic sensitivity analysis combined with accurate surrogate models, which are obtained in this paper using the D-Kriging method, is recommended for RBDO of engineering applications where accurate sensitivities of performance functions are not available. Numerical examples show the accuracy of the sensitivity results and demonstrate that sampling-based RBDO using the proposed sensitivity analysis and accurate surrogate models by the D-Kriging method yields a very accurate optimum design with good efficiency.

8. ACKNOWLEDGEMENT

Research is jointly supported by the Automotive Research Center, which is sponsored by the U.S. Army TARDEC and ARO Project W911NF-09-1-0250. These supports are greatly appreciated.

9. REFERENCES

1. Yi, K., Choi, K.K., Kim, N.H., and Botkin, M.E., "Design Sensitivity Analysis and Optimization for Minimizing Springback of Sheet-Formed Part," *Int. J. of Numerical Methods in Engineering*, Vol. 71, pp. 1483-1511, 2007.
2. Youn, B.D., Choi, K.K., and Yi, K., "Performance Moment Integration (PMI) Method for Quality Assessment in Reliability-Based Robust Optimization," *Mechanics Based Design of Structures and Machines*, Vol. 33, No. 2, pp. 185-213, 2005.
3. Youn, B.D., Choi, K.K., Yang, R-J, Gu, L., "Reliability-Based Design Optimization for Crashworthiness of Vehicle Side Impact," *Structural and Multidisciplinary Optimization*, Vol. 26, No. 3-4, pp. 272-283, 2004.
4. Youn, B.D., Choi, K.K., and Tang, J., "Structural Durability Design Optimization and Its Reliability Assessment," *International Journal of Product Development*, Vol. 1, Nos. 3/4, pp. 383-401, 2005.
5. Dong, J., Choi, K.K., Wang, A., Zhang, W., and Vlahopoulos, N., "Parametric Design Sensitivity Analysis of High Frequency Structural-Acoustic Problems Using Energy Finite Element Method," *Int. J. of Numerical Methods in Engineering*, Vol. 62, pp. 83-121, 2005.

6. Dong, J., Choi, K.K., Vlahopoulos, N., Wang, A., and Zhang, W., "Design Sensitivity Analysis and Optimization of High Frequency Radiation Problems Using Energy Finite Element and Energy Boundary Element Methods," *AIAA Journal*, Vol. 45, No. 6, pp. 1187-1198, 2007.
7. Haldar, A., and Mahadevan, S., *Probability, Reliability and Statistical Methods in Engineering Design*, John Wiley & Sons, New York, NY, 2000.
8. Madsen, H. O., Krenk, S., and Lind, N. C., *Methods of Structural Safety*, Prentice-Hall Inc., Englewood Cliffs, NJ, 1986.
9. Hasofer, A. M. and Lind, N. C., "An Exact and Invariant First Order Reliability Format," *ASCE Journal of the Engineering Mechanics Division*, Vol. 100, No. 1, pp. 111-121, 1974.
10. Ditlevsen, O., and Madsen, H.O., *Structural Reliability Methods*, John Wiley & Sons Ltd., Chichester, 1996.
11. Hohenbichler, M., and Rackwitz, R., "Improvement of Second-Order Reliability Estimates by Importance Sampling," *ASCE Journal of Engineering Mechanics*, Vol. 114, No. 12, pp. 2195-2199, 1988.
12. Breitung, K., "Asymptotic Approximations for Multinormal Integrals," *ASCE Journal of Engineering Mechanics*, Vol. 110, No. 3, pp. 357-366, 1984.
13. Rahman, S., and Wei, D., "A Univariate Approximation at Most Probable Point for Higer-Order Reliability Analysis," *International Journal of Solids and Structures*, Vol. 43, pp. 2820-2839, 2006.
14. Wei, D., "A Univariate Decomposition Method for Higher-Order Reliability Analysis and Design Optimization," Ph. D. Thesis, The University of Iowa, 2006.
15. Lee, I., Choi, K.K., Du, L., and Gorsich, D., "Inverse Analysis Method Using MPP-Based Dimension Reduction for Reliability-Based Design Optimization of Nonlinear and Multi-Dimensional Systems," *Computer Methods in Applied Mechanics and Engineering*, Vol. 198, No. 1, pp. 14-27, 2008.
16. Kalsi, M., Hacker, K., and Lewis, K., "A Comprehensive Robust Design Approach for Decision Trade-Offs in Complex Systems Design," *ASME Journal of Mechanical Design*, Vol. 123, No. 1, pp. 1-10, 2001.
17. Su, J., and Renaud, J. E., "Automatic Differentiation in Robust Optimization," *AIAA Journal*, Vol. 35, No. 6, pp. 1072-1079, 1997.
18. Rahman, S., and Xu, H., "A Univariate Dimension-Reduction Method for Multi-Dimensional Integration in Stochastic Mechanics," *Probabilistic Engineering Mechanics*, Vol. 19, pp. 393-408, 2004.
19. Xu, H., and Rahman, S., "A Generalized Dimension-Reduction Method for Multi-dimensional Integration in Stochastic Mechanics," *International Journal for Numerical Methods in Engineering*, Vol. 61, No. 12, pp. 1992-2019, 2004.
20. Lee, I., Choi, K.K., Du, L., and Gorsich, D., "Dimension Reduction Method for Reliability-Based Robust Design Optimization," *Special Issue of Computer & Structures: Structural and Multidisciplinary Optimization*, Vol. 86, pp. 1550–1562, 2008.
21. Youn, B.D. and Choi, K.K., "A New Response Surface Methodology for Reliability-Based Design Optimization," *Computers and Structures*, Vol. 82, Nos. 2-3, pp. 241-256, 2004.
22. Zhang, T., Choi, K.K., Rahman, S., Cho, K., Perry, B., and Shakil, M., and Heitka, D., "A Response Surface and Pattern Search Based Hybrid Optimization Method and Application to Microelectronics," *Structural and Multidisciplinary Optimization*, Vol. 32, No. 4, pp. 327-345, 2006.
23. Kim, C. and Choi, K.K., "Reliability-Based Design Optimization Using Response Surface Method with Prediction Interval Estimation," *ASME Journal of Mechanical Design*, Vol. 130, No. 12, pp. 1-12, 2008.
24. Rubinstein, R. Y., *Simulation and Monte Carlo Method*, John Wiley & Sons, New York, 1981.
25. Noh, Y., Choi, K.K., and Du, L., "Reliability Based Design Optimization of Problems with Correlated Input Variables Using Copulas," *Structural and Multidisciplinary Optimization*, Vol. 38, No. 1, pp. 1-16, 2009.
26. Noh, Y., Choi, K.K., and Lee, I., "Reduction of Transformation Ordering Effect in RBDO Using MPP-Based Dimension Reduction Method," *AIAA Journal*, Vol. 47, No. 4, pp. 994-1004, 2009.
27. Noh, Y., Choi, K.K., and Lee, I., "Identification of Marginal and Joint CDFs Using the Bayesian Method for RBDO," *Structural and Multidisciplinary Optimization*, Vol. 40, No. 1, pp. 35-51, 2010.
28. Rubinstein, R. Y., Shapiro A., *Discrete Event Systems – Sensitivity Analysis and Stochastic Optimization by the Score Function Method*, John Wiley & Sons, New York, 1993.
29. Rahman, S., "Stochastic Sensitivity Analysis by Dimensional Decomposition and Score Functions," Probabilistic Engineering Mechanics, Vol. 24, pp. 278-287, 2009.
30. McDonald, M., and Mahadevan, S., "Design Optimization with System-Level Reliability Constraints," *Journal of Mechanical Design*, Vol. 130, No. 2, 21403, 2008.
31. Zhao, L., Choi, K.K., Lee, I., "A Metamodel Method using Dynamic Kriging and Sequential Sampling," submitted, January, 2010.
32. Browder, A., *Mathematical Analysis: An Introduction*, Springer-Verlag, New York, 1996.
33. Hijab, O., *Introduction to Calculus and Classical Analysis*, Springer-Verlag, New York, 1997.